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Large-Density Fluctuations for the One-Dimensional Supercritical Contact Process

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We consider the one-dimensional supercritical contact process. Let T_v be the first time the process reaches a density q larger than the equilibrium one ρ in the region $[1...N]$. We prove that, starting from equilibrium, $T_N/E(T_N)$ converges to an exponential random time of mean one. In this way we extend previous results of Lebowitz and Schonmann.

KEY WORDS: Infinite-particle systems; random dynamics; large deviations.

1. INTRODUCTION

Here we study the asymptotic distribution of the time of first occurrence of an anomalous density of particles in a large, fixed region of the space for the supercritical one-dimensional contact process. The contact process is a Markovian system with infinitely many particles in $\mathbb{Z}$, the set of all integers. The system evolves in the following way: each site of $\mathbb{Z}$ may be either empty or occupied by at most one particle. Each particle, independently of the others, waits for an exponential random time with mean $1/(2\lambda+1)$, where $\lambda > 0$, and then decides to die with probability $1/(2\lambda+1)$ or to put a new particle at the first site on the right (respectively on the left) with probability $\lambda/(2\lambda+1)$ [respectively $\lambda/(2\lambda+1)$]. If the particle decides to put a new one on a site which was already occupied, nothing happens. After each such choice the procedure starts again, independently of the past.

As was first shown by Harris,[10] there is a critical value λ_c, with $0 < \lambda_c < \infty$, such that if $\lambda < \lambda_c$, then the only invariant probability measure

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for the process is the Dirac δ-measure concentrated in the empty configuration. On the other hand, in the supercritical case when $\lambda > \lambda_c$, there is another extremal invariant probability measure ν whose support is contained in the set of the configurations having infinitely many particles with a given density ρ, which, of course, depends on λ.

Here we are interested in the time needed by the process to have an anomalous density of particles q, with $q > \rho$ in the set $\{1, 2,..., N\}$. We show that, starting with the invariant probability measure ν, then, when suitably rescaled, this time converges in law for $N \to \infty$ to a random exponential time with mean one. Moreover, we show that the scaling factor is logarithmically equivalent to $\exp[N\phi(q)]$, where ϕ: $[0, 1] \to [0, \infty]$ is the function introduced in refs. 6 and 7 and studied in more detail in ref. 3. Actually, our proof works also for the case in which the aomalous density is smaller than ρ, but this was already done in ref. 7 for a large class of attractive, infinitely-many-particle systems, including the contact process.

The point is that in ref. 7 the proofs are all based on the following monotonicity property due to attractiveness: if the process starts with the configuration in which all sites are occupied, then during the time evolution its distribution decreases toward the invariant probability measure. This monotonicity can be used to study the time it takes for the process to reach a density smaller than ρ, but just does not work if the fluctuations are in the "wrong" upper direction.

Here we use a different approach based on the quick loss of memory of the process. This follows from our basic lemma, which says that starting with two different configurations with enough particles and letting them evolve with the same choise of deaths and creations, then very quickly they become identical forever in any fixed region of the space. By "quickly" here we mean "much smaller than the time needed to perform the large density fluctuation." This approach was implicit in the papers which studied the so-called pathwise approach to metastability (see e.g. [CGOV]) and it was put in evidence and intensively used in Refs. 2, 8, and 9.

We would like to stress that this work was much inspirated by Ref. 7, where the reader can find a very interesting discussion about occurrence times of rare events for infinite particles systems.

In Section 2 we introduce the contact process, give the basic definitions, and state the theorem. In Section 3 we state and prove our basic lemma and give the proof of the theorem.

2. NOTATIONS AND RESULTS

The contact process can be constructed in the following way: for $x \in \mathbb{Z}$, let $(U_n^{(x,x+1)}, n \geq 1)$, $(U_n^{(x,x-1)}, n \geq 1)$, and $(U_n^{+,x}, n \geq 1)$ be mutually independent Poisson point processes in $\mathbb{R}^+$ with intensity λ, λ, and 1, respectively. We also suppose that the Poisson processes corresponding to different values of x are mutually independent and we denote by $(\Omega, \mathbf{P})$ the probability space where all such Poisson processes are defined.

Definition. Given $t>0$, x and y in $\mathbb{Z}$, and ω in Ω, we say that there is an *ω-path* from $(x, 0)$ to (y, t) if there is a finite sequence of points $x_0, x_1,..., x_k$ with $x_0=x$, $x_k=y$, and $|x_i-x_{i+1}|=1$, for $i=0, 1, k-1$, and there are integers n_1, I, n_k such that

$$0<U_{n_1}^{(x_0, x_1)}(\omega)< \cdots <U_{n_k}^{(x_{k_1}, x_k)}(\omega)<t \tag{2.1}$$

and for no j and m

$$0\leqslant U_m^{+,x}(\omega)\leqslant U_{n_1}^{(x,x_1)}(\omega)$$

$$U_{n_{j-1}}^{(x_{j-1}, x_j)}(\omega)\leqslant U_m^{+,x_j}(\omega)\leqslant U_{n_j}^{(x_j, x_{j+1})}(\omega)$$

$$U_{n_k}^{(x_{k-1}, y)}(\omega)\leqslant U_m^{+,y}(\omega)\leqslant t$$

Given a configuration $\eta\in\{0, 1\}^{\mathbb{Z}}$, $t>0$, and $\omega\in\Omega$, we define the configurations $\xi_t^\eta(\omega)\in\{0, 1\}^{\mathbb{Z}}$ in the following way: for any $y\in\mathbb{Z}$, $\xi_t^\eta(\omega, y)=1 \Leftrightarrow$ there is an $x\in\mathbb{Z}$ such that $\eta(x)=1$ and there is an ω-path from $(x, 0)$ to (y, t).

The Markov process $(\xi_t^\eta)_{t\geqslant 0}$ is what we call the contact process starting with configuration η at time 0. We shall write ξ_t^η for the evolution at time t of the process starting with just one particle at the site $a\in\mathbb{Z}$, that is, $\xi_0^a(y)=1$ only for $y=a$. In the case $\lambda>\lambda_c$ this process has two extremal invariant probability measures: the Dirac measure concentrated in the empty configuration and ν, which is a nontrivial measure which can be defined in the following way: for any finite set $F\subset\mathbb{Z}$

$$\nu(\eta;\eta(x)=0 \text{ for } x\in F)=\mathbf{P}(\tau^x<\infty, x\in F) \tag{2.2}$$

where $\tau^x=\inf(t>0;\ \xi_t^x(y)=0, \forall y\in\mathbb{Z})$.

The support of ν is contained in the set of all the configurations having density ρ where $\rho=\mathbf{P}(\tau^x=\infty)$.

For more details about the construction of the contact process and its main properties we refer the reader to Chapter 6 of ref. 5.

Let

$$B(N, a, b)=\left\{\eta\in\{0\ 1\}^{\mathbb{Z}}:\quad aN\leqslant\sum_{x=1}^{N}\eta(x)\leqslant bN\right\} \tag{2.3}$$

where $0 < a < b < 1$. In refs. 3 and 6 it was shown that there exists a convex function ϕ: $[0, 1] \to [0, +\infty]$ such that $\phi(p) = 0$ if an only if $p = \rho$ and

$$\lim_{N \to \infty} \frac{1}{N} \log[\nu(B(N, a, b))] \leftrightarrow -\inf_{a \leqslant p \leqslant b} \phi(p) \tag{2.4}$$

For any $q > \rho$ and N positive integer let

$$A(N, q) = \left\{ \eta \in \{0, 1\}^{\mathbb{Z}};\ \sum_{x=1}^{N} \eta(x) > qN \right\} \tag{2.5}$$

We also define $T_N^\eta(q)$ as the first time the process ξ_t^η reaches the set $A(N, q)$, i.e.,

$$T_N^\eta(q) = \inf\{t > 0;\ \xi_t^\eta \in A(N, q)\} \tag{2.6}$$

For notational convenience we will write $T_N^\eta = T_N^\eta(q)$ whenever no confusion is possible.

If the initial configuration is choosen at random with the invariant distribution ν, then we will simply write $T_N^\nu(q)$ or T_N^ν whenever no confusion is possible. Finally, let β_N be implicitly defined by

$$\mathbf{P}(T_N^\nu > \beta_N) = e^{-1} \tag{2.7}$$

Theorem. For any $q > \rho$:

1. $T_N^\nu(q)$ converges in law as $N \to \infty$ to a mean one exponential random time.
2. $\lim_{N \to \infty} (1/N) \log(\beta_N) = \phi(q)$.
3. $\lim_{N \to \infty} [E(T_N^\nu)/\beta_N] = 1$.

3. PROOF OF THE THEOREM

The proof of the theorem uses two main ingredients. The first one is the quick loss of memory of the process, which is the content of the Basic Lemma. The second one is an *a priori* lower bound of the time of occurrence of a rare event. This is done in Proposition 1.

Basic Lemma. For any p: $1 < p < 0$ there exists positive constants c and k such that for every $\eta, \varsigma \in A(N, p)$

$$\mathbf{P}(\xi_t^\eta(x) = \xi_t^\varsigma(x), \quad x = 1,..., N; \quad \forall t > kN) \geqslant 1 - e^{-cN} \qquad \forall N \geqslant 1 \tag{3.1}$$

Proof. It is enough to prove the result for $\eta \in A(N, p)$ and ς such that $\varsigma(x) = 1$, $\forall x$.

Since $\sum_{x=1}^{N} \eta(x) \geqslant pN$, we have, using Theorem (3.29), Chapter 6, ref. 5,

$$\mathbf{P}(\exists x \in [1, N] \cap \mathbb{Z}; \eta(x) = 1 \quad \text{and} \quad \tau^x = +\infty) \geqslant 1 - \exp(-cpN) \tag{3.2}$$

Let now x be the leftmost of those particles in η such that $\tau^x = +\infty$ and call, for any $t > 0$,

$$l_t^x = \min\{y; \xi_t^x(y) = 1\}; \qquad r_t^x = \max\{y; \xi_t^x(y) = 1\}$$

It is very easy to show (see, e.g., Theorem 2.2, Chapter 6, ref. 5) that $\eta(x) = 1$ and $\tau^x = +\infty$ imply

$$\xi_t^\eta(y) = \xi_t^x(y)$$

for any $l_t^x \leqslant y \leqslant r_t^x$ and any t.

Therefore the lemma is proved if we can show that

$$\mathbf{P}([1, N] \cap \mathbb{Z} \subset [l_t^x, r_t^x] \quad \forall t > kN) \geqslant 1 - \exp(-cN) \tag{3.3}$$

This follows immediately if k is taken large enough from Corollary 3.2, Chapter 6, ref. 5.

Proposition 1. Let D_N be any cylindrical event depending only upon the coordinates $x = 1,..., N$ such that

$$v(D_N) \leqslant \frac{c}{N^{2(1+\delta)}}$$

for some constants $c \geqslant 1$, $\delta > 0$. Let $\sigma_N = \inf\{t; \xi_t^v \in D_N\}$. Then

$$\lim_{N \to \infty} \mathbf{P}(\sigma_N < N^{1+\delta}) = 0$$

Proof. We have

$$\begin{aligned}\mathbf{P}(\sigma_N < t \leqslant \mathbf{P}\left(\frac{\xi_k^v}{N^{1+\delta}} \in D_N \text{ for some } k = 1 \cdots [tN^{1+\delta}] + 1\right) \\ + \mathbf{P}\left(\sigma_N < t; \xi_{k/N^{1+\delta}}^v \notin D_N \text{ for any } k = 1 \cdots [tN^{1+\delta}] + 1\right)\end{aligned} \tag{3.4}$$

The first term is smaller than

$$(tN^{1+\delta} + 1)\, v(D_N) \leqslant (tN^{1+\delta} + 1) \frac{c}{N^{2(1+\delta)}} \leqslant \frac{2ct}{N^{1+\delta}} \tag{3.5}$$

while the second one is estimated from above by the probability that one among the $3N$ independent Poisson point processes associated with the sites $x = 1, \ldots, N$ fires between $k/N^{1+\delta}$ and $(k+1)/N^{1+\delta}$, where k is such that

$$\sigma_N \in \left[\frac{k}{N^{1+\delta}}, \frac{k+1}{N^{1+\delta}}\right]$$

Since σ_N is a stopping time of the global Poisson point process obtained by putting together all the poisson processes attached to each site $x \in \mathbb{Z}$, the probability of the above event can be bounded by

$$1 - \exp\left(-\frac{N(2\lambda+1)}{N^{1+\delta}}\right)$$

which for large N is bounded by c'/N^{δ} for some $c' > 0$.

In conclusion,

$$\mathbf{P}(\sigma_N < t) \leqslant \frac{2ct}{N^{1+\delta}} + \frac{c'}{N^{\delta}} \tag{3.6}$$

which goes to zero if $t \leqslant N^{1+\delta/2}$.

We now turn to the proof of the first part of the theorem. It is enough (see, e.g., ref. 1) to show that

$$\lim_{N \to \infty} [\mathbf{P}(T_N^{\nu} > \beta_N(t+s)) - \mathbf{P}(T_N^{\nu} > \beta_N t)\,\mathbf{P}(T_N^{\nu} > \beta_N s)] = 0$$

We remark that:

$$\lim_{N \to \infty} |\mathbf{P}(T_N^{\nu} > \beta_N(t+s)) - \mathbf{P}(\xi_u^{\nu} \notin A(N, q),$$
$$u \in [0, \beta_N t] \cup [\beta_N t + N^{1+\delta}, \beta_N(t+s)])| = 0$$

and

$$\lim_{N \to \infty} |\mathbf{P}(T_N^{\nu} > \beta_N s) - \mathbf{P}(\xi_u^{\nu} \notin A(N, q), u \in [N^{1+\delta}, \beta_N s])| = 0$$

In fact,by the strong Markov property and the invariance of ν, these differences are both upper-bounded by $\mathbf{P}(T_N^{\nu} < N^{1+\delta})$, which goes to zero by Proposition 1. Therefore, it is enough to prove that

$$\lim_{N \to \infty} |\mathbf{P}(\xi_u^{\nu} \notin A(N, q), u \in [0, \beta_N t] \cup [\beta_N t + N^{1+\delta}, \beta_N(t+s)])$$
$$- \mathbf{P}(T_N^{\nu} > \beta_N t)\,\mathbf{P}(\xi_u^{\nu} \notin A(N, q), u \in [N^{1+\delta}, \beta_N s])| = 0$$

Using the Markov property, we write

$$\mathbf{P}(\xi_u^\nu \notin A(N, q), u \in [0, \beta_N t] \cup [\beta_N t + N^{1+\delta}, \beta_N(t+s))$$
$$= \int \mathbf{P}(T_N^\nu > \beta_N t, \xi_{\beta_N t}^\nu \in d\eta) \cdot \mathbf{P}(\xi_u^\eta \notin A(N, q), u \in [N^{1+\delta}, \beta_N s])$$

We remark that $\mathbf{P}(T_N^\nu > \beta_N t,\ \xi_{\beta_N t}^\nu \notin B(N, a, b)) \leqslant \nu(B(N, a, b))$, which goes to zero as $N \to \infty$ if $0 < a < \rho < b$.

Therefore, it is enough to show that

$$\sup_{\zeta, \eta \in B(N, a, b)} |\mathbf{P}(\xi_u^\zeta \notin A(N, q), u \in [N^{1+\delta}, \beta_N s])$$
$$- \mathbf{P}(\xi_u^\eta \notin A(N, q), u \in [N^{1+\delta}, \beta_N s]| \to 0 \qquad \text{as} \quad N \to \infty$$

This supremum is upper-bounded by

$$\sup_{\zeta, \eta \in B(N, a, b)} \mathbf{P}(\xi_u^\eta(x) \neq \xi_u^\zeta(x) \qquad \text{for some} \quad x \in \{1, 2, ..., N\} \text{ and}) \\ \text{some } u > N^{1+\delta} \tag{3.7}$$

which goes to zero as $N \to \infty$ by the Basic Lemma.

This concludes the proof of the first part of the theorem.

We now prove the second part of the theorem. We will follow the pattern of Theorem 2 of ref. 7.

Let $\gamma > \phi(q)$. We will show that

$$\mathbf{P}(T_N^\nu > e^{\gamma N}) \to 0 \qquad \text{as} \quad N \to \infty \tag{3.8}$$

This implies that $\beta_N < e^{\gamma N}$. On the other hand, using (3.6), we have

$$\mathbf{P}(T^\nu < t_0) \leqslant (t_0 N^{\delta+1} + 1)\, \nu(A(N, q)) + \frac{c'}{N^\delta}\ \forall \delta > 0 \tag{3.9}$$

Therefore, since $\lim_{N \to \infty} \{\log[\nu(A(N, q)]/N = -\phi(q) < 0$, by Theorem 1 of ref. 3 we have that if $t_0 < e^{[\phi(q) - \delta]N}$ for some fixed $\delta > 0$, then

$$\mathbf{P}(T^\nu < t_0) \to 0 \qquad \text{as} \quad N \to \infty$$

Thus $\beta_N > e^{[\phi(q) - \delta]N}$ for any large enough.

We now turn to the proof of (3.8). We have

$$\mathbf{P}(T_N^\nu > e^{\gamma N}) \leqslant \mathbf{P}(T_N^\nu > e^{\gamma N};\ \sigma_\delta^\nu > e^{\gamma N}) + \mathbf{P}(\sigma_\delta^\nu < e^{\gamma N}) \tag{3.10}$$

where

$$\sigma_\delta^\nu = \inf\left\{t\colon \quad \sum_{x=0}^{[N^{3/2}]} \xi_t^\nu(x) < N^{3/2}\delta\right\}$$

with $\delta < \rho$. Using Proposition 1, we get that

$$\mathbf{P}(\sigma_\delta^\nu = e^{\gamma N}) \to 0 \qquad \text{as} \quad N \to \infty$$

since

$$\nu\left(\xi\colon \quad \sum_{x=0}^{[N^{3/2}]} \xi_t^\nu(x) < N^{3/2}\delta\right) \leqslant \exp[-\phi(\delta)\, N^{3/2}/2]$$

for N large enough. To estimate the first term, we write

$$\begin{aligned}
&\mathbf{P}(T_N^\nu > \exp(\gamma N);\, \sigma_\delta^\nu > \exp(\gamma N)) \\
&\quad \leqslant \mathbf{P}\left(\xi_{kN^2}^\nu \notin A(N, q);\, \xi_{kN^2}^\nu \notin A(N^{3/2}, \delta), \quad \forall k = 1 \cdots \left[\frac{\exp(\gamma N)}{N^2}\right]\right) \\
&\quad \leqslant \{\mathbf{P}(\xi_{kN^2}^\nu \notin A(N, q)) + \exp(-cN^{3/2})\}[\{\exp(\gamma N)\}/N^2] \qquad (3.11)
\end{aligned}$$

where we have used the basic lemma and the fact that $N^2 \gg N^{3/2}$ for large N. Since

$$\mathbf{P}(\xi_{kN^2}^\nu \notin A(N, q)) = 1 - \nu(A(N, q)) \leqslant 1 - e^{-[\phi(q) + \delta']N}$$

$\forall \delta' > 0$ and N large enough, we get (3.8).

This concludes the proof of the second part. To prove the third statement, first we write

$$\frac{\mathbf{E}(T_N^\nu)}{\beta_N} = \int_0^\infty dt\, \mathbf{P}(T_N^\nu > \beta_N t)$$

In order to perform the limit $N \to \infty$ inside the integral, we will show that there exists $h(t) \geqslant 0$ with $\int_0^\infty dt\, h(t) < \infty$ and

$$\mathbf{P}(T_N^\nu > \beta_N t) < h(t)$$

for any N large enough. Let

$$C_t^{k,e} = \left\{\eta\colon \quad \sum_{x=-\sqrt{t}}^{kN + [\sqrt{t}]} \eta(x) < \varepsilon\right\}$$

and let

$$S_t^\nu = \inf\{s\colon \quad \xi_s^\nu \in C_t^{k,\varepsilon}\}$$

Using (3.9) and again Theorem 1 of ref. 3, we see that $\mathbf{P}(S_t^\nu < \beta_N t)$ is bounded by

$$(\beta_N t N^{1+\delta} + 1)\, \nu(C_t^{k,\delta}) + \frac{c'\delta}{kN + 2\sqrt{t}}$$

and this last expression, if k and δ are large enough, can be bounded by an L^1 function $h_1(t)$ independent of N. Therefore

$$\begin{aligned}\mathbf{P}(T_N^\nu > \beta_N t) &\leqslant \mathbf{P}(T_N^\nu > \beta_N t;\, S_t^\nu > \beta_N t) + \mathbf{P}(S_t^\nu < \beta_N t)\\ &\leqslant \int \mathbf{P}(T_N^\nu > \beta_N(t - c\sqrt{t});\, S_t^\nu > \beta_N t;\, \xi^\nu_{\beta_N(t-c\sqrt{t}} \in d\eta)\\ &\quad \times \mathbf{P}(T_N^\eta > \beta_N c\sqrt{t}) + h_1(t) \end{aligned} \tag{3.12}$$

Again using Proposition 1, we have that

$$\sup_{\eta \notin C_t^{k,\varepsilon}} |\mathbf{P}(T_N^\eta > \beta_N c\sqrt{t}) - \mathbf{P}(T_N^\nu > \beta_N c\sqrt{t})| \to 0 \qquad \text{as} \quad N \to \infty$$

if c large enough. Thus, if $c\sqrt{t} \gg 1$,

$$\sup_{\eta \notin C_t^{k,\varepsilon}} \mathbf{P}(T_N^\eta > \beta_N c\sqrt{t}) < e^{-1}$$

which implies, together with (3.12), that

$$\mathbf{P}(T_N^\nu > \beta_N t;\, S_t^\nu > \beta_N t) < \exp(-c\sqrt{t})$$

if $c\sqrt{t} \gg 1$.

In conclusion,

$$\mathbf{P}(T_N^\nu > \beta_N t) \leqslant \exp(-t^{1/2}/c) + h_1(t) = h(t)$$

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